Relationships Among Derivatives

1 Suppose we have a function $\rho(x(t),t)$. What is the general expression for $\frac{d\rho}{dt}$? If $\rho = e^{x^2+i\omega t}$ and $x(t) = cos(\alpha t+\phi)$, find both $\frac{\partial \rho}{\partial t}$ and $\frac{d\rho}{dt}$. Discuss how we can interpret the differences between the partial and total derivative.

2 Suppose that$f(x,y) = \frac{\cos x}{\ln(y)}$. Show the statement below holds and discuss a geometric interpretation.

$$\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_x\right)_y = \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_y\right)_x$$

3 Imagine there exists some function $f(x,y,z)$, and we want to take a constant slice of the function such that $f(x,y,z) = C$. This process specifies a relationship between the variables x, y, and z. And in fact, regardless of what f is, we can make a precise statement about the derivatives of x, y, and z with respect to one another.

(a) First write down the total differentials of x and y on this constant slice of f. Your answer should look like $dx =$ [something in terms of dy and dz] and $dy =$ [something in terms of dx and dz].

(b) Now eliminate dy from the equations above to get a relationship between dx and dz.

(c) Finally, show that this equation implies the triple product rule

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1$$

You should make use of the fact that in general $\left(\frac{\partial g}{\partial h}\right)_\alpha\left(\frac{\partial h}{\partial g}\right)_\alpha = 1$ for variables involved in this setup.

Applying Integration

4 Suppose that molecule A takes the shape $A(x,y,z)$ in cartesian coordinates and molecule B is described by $B(r,\theta,\phi)$ in spherical coordinates. Set up the integrals to find the following:

(a) The total volume of molecule B.

(b) The volume of molecule B contained within a cone of 30° extending from the origin around the vertical axis.

(c) The cross section of molecule A in the x-y plane. (Hint: this should be a function of z.)

(d) How would you go about calculating part b) for molecule A, and part c) for molecule B? Do not set up the integrals, but discuss the additional steps required and the advantages of using certain coordinate systems.

5 Use the equality below

$$\int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

to compute the following integral

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx.$$

6 Using the integration by parts formula $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$, integrate

$$\int e^x \cos x \, dx$$

Your answer should not have an integral in it.

7 **Bonus problem:** Suppose we make the definition

$$\mathcal{L}[f(x)] = -\frac{d}{dx}\big(\mu(x)f(x)\big) + D\frac{d^2 f}{dx^2}.$$

($\mathcal{L}$ is called the Fokker-Planck operator, which might be relevant to you at some point, but is irrelevant for solving this problem.) Note that μ is a function of x, but you don't need to know its form for this problem. D is just a number. If we want to define $\mathcal{L}^\dagger$ such that for two differentiable functions $f(x)$ and $g(x)$, it's always true that

$$\int_{-\infty}^{\infty} \mathrm{d}x \; g(x)\mathcal{L}[f(x)] = \int_{-\infty}^{\infty} \mathrm{d}x \; f(x)\mathcal{L}^\dagger[g(x)],$$

show that

$$\mathcal{L}^\dagger[f(x)] = \mu(x)\frac{df}{dx} + D\frac{d^2 f}{dx^2}$$

You will need to assume that f(x), g(x), and all of their derivatives go to zero at both positive and negative infinity. In addition, something of the form $f(x)g(x)\mu(x)$ is zero at postive and negative infinity.