Transformations

1. Determine which (if any) of the following operators correspond to linear transformations on some arbitrary function $f(x)$:

   (a) $\hat{A} = \frac{d}{dx}$

   (b) $\hat{B} = \frac{d}{dx}x$

   (c) $\hat{C} = \log(\cdot)$

   (d) $\hat{D} = \int dx$

2. Consider an operator $\hat{O}$ that acts on a four-dimensional vector space with basis functions $|\phi_k\rangle$ for $k = 1, 2, 3$, and 4. Somehow you know the operator acts on each of the basis functions according to

$$\hat{O}\,|\phi_1\rangle = i\,|\phi_2\rangle + 3\,|\phi_3\rangle \tag{1}$$

$$\hat{O}\,|\phi_2\rangle = -i\,|\phi_1\rangle - 2i\,|\phi_3\rangle \tag{2}$$

$$\hat{O}\,|\phi_3\rangle = 3\,|\phi_1\rangle + 2i\,|\phi_2\rangle + |\phi_3\rangle \tag{3}$$

$$\hat{O}\,|\phi_4\rangle = \frac{\pi}{2}\,|\phi_4\rangle \tag{4}$$

   (a) Assuming these four basis functions form an orthonormal set ($\langle\phi_i|\phi_j\rangle = \delta_{ij}$), write the matrix representation for the operator $\hat{O}$ in this basis.

   (b) Is it possible for this operator to correspond to a physical observable? What could one say about the eigenvalues of $\hat{O}$?

   (c) Determine the action of this operator on the vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle - |\phi_3\rangle\right) \tag{5}$$

   (d) BONUS: Suppose you find a new basis $\{|\chi\rangle\}$ where $|\chi_1\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle + |\phi_2\rangle\right)$, $|\chi_2\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle - |\phi_2\rangle\right)$, $|\chi_3\rangle = |\phi_3\rangle$, and $|\chi_4\rangle = |\phi_4\rangle$. Show that this new basis also forms and orthonormal set. Then, write the matrix representation of $\hat{O}$ in this new basis.

Classes of Operators

3. In the notes (Example 5.1) we found the following form for a matrix that rotates a vector by some angle θ about the y-axis:

$$\mathcal{R}(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}. \tag{6}$$

   Show that this matrix is a unitary matrix. If one applies $\mathcal{R}(\theta)$ to an arbitrary vector, what happens to the norm $\|\cdot\|$ of that vector?

4. Let $|a\rangle$, $|b\rangle$, and $|c\rangle$ be vectors defined in terms of the Cartesian basis vectors according to

$$|a\rangle = \frac{1}{\sqrt{2}}|x\rangle + \frac{1}{\sqrt{2}}|y\rangle \tag{7}$$

$$|b\rangle = \frac{1}{\sqrt{2}}|x\rangle - \frac{1}{\sqrt{2}}|y\rangle \tag{8}$$

$$|c\rangle = |z\rangle\,. \tag{9}$$

Let the vector space V consist of the span of $|a\rangle$ and $|b\rangle$ and the vector space W of the span of $|a\rangle$ and $|c\rangle$.

(a) Write the operators for the projection onto each of these two vector spaces, V and W. Recall that for a normalized basis, $\hat{P}_V = \sum_{i=1}^{N} |\phi_i\rangle\langle\phi_i|$.

(b) Determine the matrix representation for each of these two operators.

(c) Using either the operator you obtained in (a) or the matrix you obtained in (b), obtain the projection of

$$|\omega\rangle = \frac{1}{\sqrt{3}}\left(|x\rangle + |y\rangle + |z\rangle\right)$$

onto each of the vector spaces V and W.