Orthonormal Basis Vectors

Consider these vectors for the following questions:

$$|A\rangle = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad |B\rangle = \begin{bmatrix} i \\ 2-i \end{bmatrix} \qquad |C\rangle = \begin{bmatrix} 0 \\ i \end{bmatrix} \qquad |D\rangle = \begin{bmatrix} 2i-1 \\ -1 \end{bmatrix}$$

1. Write the *bra* $\langle\cdot|$ corresponding to each *ket* $|\cdot\rangle$.
2. Which pair (only one) of vectors above is orthogonal?
3. What is the span this pair of vectors?
4. Normalize the orthogonal pair of vectors you found in (2) above. This makes the pair *orthonormal*.
5. Write all other vectors in (2) that were not orthogonal to any other vectors as a linear combination of the orthonormal vector pair from(4).
6. Express your results from (5) as vectors in this new orthonormal basis.

Function Spaces

7. For this problem, we will be considering functions defined over the region $x \in [0, 1]$. Mathematically, we will have

$$f(x) = \begin{cases} g(x) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} ,$$

where $g(x)$ is some other function. Within this function space, we'll work with basis functions defined according to

$$g_k(x) = \sin(k\pi x) \qquad k = 1, 2, 3, \ldots .$$

(a) What is the definition of the inner product for this function space?

(b) Choose any two functions from this set and show that they are orthogonal.

(c) Normalize the function corresponding to $k = 2$.

Inner Product Properties

8. The Cauchy-Schwarz inequality can be written as follows:

$$|\langle\phi|\psi\rangle|^2 \leq \langle\phi|\phi\rangle \langle\psi|\psi\rangle .$$

Show that in the case where $|\phi\rangle$ and $|\psi\rangle$ are linearly dependent ($|\phi\rangle = a|\psi\rangle$ for $a \in \mathbb{C}$) this is actually an equality.