1. **Key Properties of the Fourier Transform**. For notation, let $f(t)$ be some arbitrary function in the time domain and $\hat{f}(\omega)$ be its Fourier transform.

   (a) Show that the Fourier transform is a linear transformation. Review your notes from Linear Algebra part 3 for the definition of linearity.

   (b) Show that shifting the function in time by t_0 results in a phase factor in its Fourier transform. Specifically, show that $\mathcal{F}\{f(t-t_0)\} = c \cdot \hat{f}(t)$, and find the constant, c.

   (c) Suppose that f is a real function. Show that complex-conjugating its Fourier transform $\hat{f}$ is the same as horizontally inverting that transform, that is, $\left(\hat{f}(\omega)\right)^\dagger = \hat{f}(-\omega)$

   (d) Fourier transforms also have applications when manipulating the derivatives of functions. For a smooth function, f, find the Fourier transform of $\frac{\mathrm{d}f(t)}{\mathrm{d}t}$ in terms of the Fourier transform of f. Can you generalize to $\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$?

2. Write $g(x) = x/\pi$, defined as a periodic function within $x \in [-\pi, \pi)$, as a real Fourier series with sines and cosines. Make use of the orthogonality of the sine and cosine bases, and the fact that $g(x)$ is an odd function. A plot of $g(x)$, the 'sawtooth wave', is here.

   Use Desmos, an online graphing calculator, to plot successive Fourier approximations to $g(x)$.

   The orthogonality relations for the Fourier basis are as follows for $m, n \neq 0$,

$$\int_{-\pi}^{\pi} dx \sin(mx)\sin(nx) = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{1}$$

$$\int_{-\pi}^{\pi} dx \cos(mx)\cos(nx) = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2}$$

$$\int_{-\pi}^{\pi} dx \sin(mx)\cos(nx) = 0 \tag{3}$$

$$\int_{-\pi}^{\pi} dx \sin(mx) = \int_{-\pi}^{\pi} dx \cos(mx) = 0. \tag{4}$$

3. Obtain the Fourier transform of the following functions

   (a) The Gaussian function

$$f(t) = e^{-at^2}. \tag{5}$$

   How does the Fourier transform change graphically when a is increased? You may use a graphing utility like Desmos to plot.

(b) The Dirac delta function, which is defined as

$$\delta(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases} \tag{6}$$

such that

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1.$$

Looking at the properties for this function on Wolfram might be useful. Based on your answer, what is the Fourier transform of a planewave at a fixed frequency ω_0, that is, $f(t) = e^{i\omega_0 t}$?

4. Bonus: We define the *convolution* of two functions f and g as

$$h(t) \equiv (f * g)(t) \equiv \int_{-\infty}^{\infty} f(t - t')g(t')dt'. \tag{7}$$

Express the Fourier transform of h in terms of the Fourier transforms of f and g.